Discrete Applied Mathematics 5 (1983) 241–242
North-Holland Publishing Company

NOTE

A CONVEX POLYTOPE OF DIAMETER ONE

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Received 9 June 1980
Revised 30 March 1981

In this note, we give an example of a convex polytope in $\mathbb{R}^{n^2}$, with 2^n extreme points, which has diameter one. This polytope is related to the quadratic pseudo-Boolean problem $\mathrm{Max}_{X\in B^n} X'QX$ where $B=\{0,1\}$ and $Q=(q_{ij})$ is a $n\times n$ real symmetric matrix.

If we associate with any $X\in B^n$

$$p(X)=(x_1x_1,x_1x_2,\ldots,x_1x_n,x_2x_1,x_2x_2,\ldots,x_nx_{n-1},x_nx_n)\in B^{n^2},$$

then the above mentioned quadratic optimisation problem is equivalent to the linear program $\mathrm{Max}_{y\in C}\, q'y$ where

$$q=(q_{11},q_{12},\ldots,q_{1n},q_{21},q_{22},\ldots,q_{nn}),$$

and

$$C=\text{convex hull of } \{p(X)\mid X\in B^n\}.$$

We prove that C, which is a convex polytope in $\mathbb{R}^{n^2}$ with 2^n extreme points, has diameter one. Thus any method which depends upon searching the adjacent extreme points of a given extreme point is equivalent to exhaustive enumeration.

Theorem. *If $X^1\in B^n, X^2\in B^n, X^1\neq X^2$, then $p(X^1)$ and $p(X^2)$ are adjacent extreme points on C.*

Proof. We will use the following notations:

$$J(X)=\{i\mid x_i=1\},\quad X\in B^n,$$
$$S^1=J(X^1)\cap J(X^2),\qquad S^2=J(X^1)-J(X^2),\qquad S^3=J(X^2)-J(X^1),$$
$$|S^i|=s_i,\quad i=1,2,3\qquad\text{and}\qquad |J(X^1)|=k.$$

Clearly, the roles of X^1 and X^2 can be interchanged, if necessary, to ensure that $|J(X^1)|\le|J(X^2)|$.

Let $f=(f_{11},f_{12},\ldots,f_{21},f_{22},\ldots,f_{nn})\in\mathbb{R}^{n^2}$ be constructed as follows:

$$f_{ij}=\begin{cases}1/2 & \text{if } i,j\in J(x^1),\\ s_2/(2s_3) & \text{if } i\in S^1, j\in S^3;\ or\ i\in S^3, j\in S^1,\\ 1+\dfrac{s_2(s_2-1)}{2s_3(s_3-1)} & \text{if } i,j\in S^3,\\ m<-4k^2 & \text{otherwise}\end{cases} \qquad i\neq j$$

$$f_{ii}=\begin{cases}1 & \text{if } i\in J(X^1),\\ 1-s_3+s_2/s_3 & \text{if } i\in S^3,\\ m<-4k^2 & \text{otherwise.}\end{cases}$$

The vector f obviously depends upon X^1 and X^2. We show that $H=\{Y \mid f'Y=\frac{1}{2}k(k+1)\}$ is a supporting hyperplane of C such that $H\cap C=\{p(X^1), p(X^2)\}$.

It is obvious that $f'p(X^1)=\frac{1}{2}k(k+1)$. Further,

$$\begin{aligned}f'p(X^2)&=s_1+(1-s_3+s_2/s_3)s_3\\&\quad+\left\{\tfrac{1}{2}s_1(s_1-1)+2s_1s_3\frac{s_2}{2s_3}+s_3(s_3-1)\left[1+\frac{s_2(s_2-1)}{2s_3(s_3-1)}\right]\right\}\\&=\tfrac{1}{2}k(k+1).\end{aligned}$$

Also if X $(\neq X^1, X^2)\in B^n$, then $f'p(X)<\frac{1}{2}k(k+1)$ provided $J(X)\subset J(X^1)$ or $J(X)\subset J(X^2)$. If $J(X)\not\subseteq J(X^1)$ and $J(X)\not\subseteq J(X^2)$, then in view of the construction of f, we must have $f_{ij}x_ix_j=m$ for some i and j. Since the sum of positive components of f cannot exceed $4k^2$, we conclude that $f'p(X)<0<\frac{1}{2}k(k+1)$. This completes the proof of the theorem.

As pointed out by one of the referees, the above theorem can be restated as: "Given $Y, Z\in B^h$, $Y\neq Z$, there exists a $n\times n$ symmetric matrix A such that $Y'AY=Z'AZ$ and $Y'AY>X'AX$ for all X $(\neq Y, Z)\in B^n$".

Acknowledgement

We are thankful to the referees for their constructive suggestions.